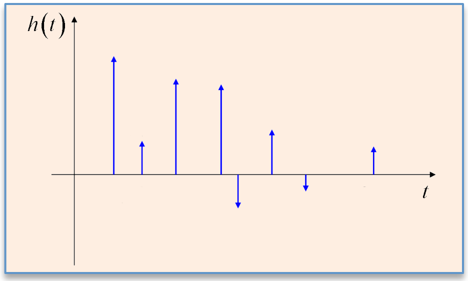
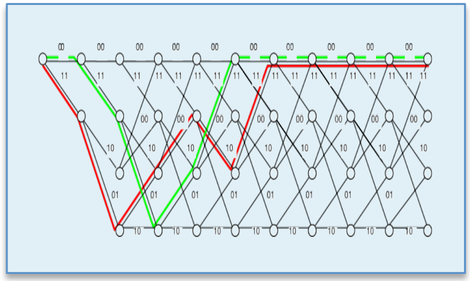
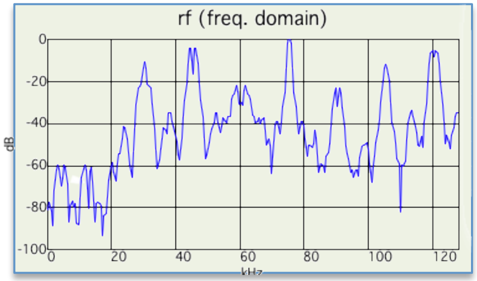
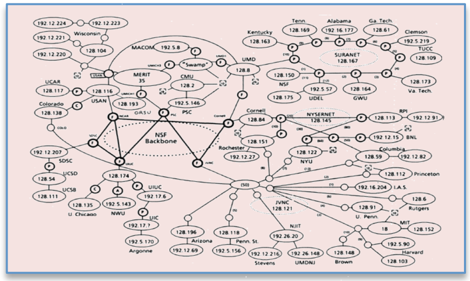








INTRODUCTION TO EECS II

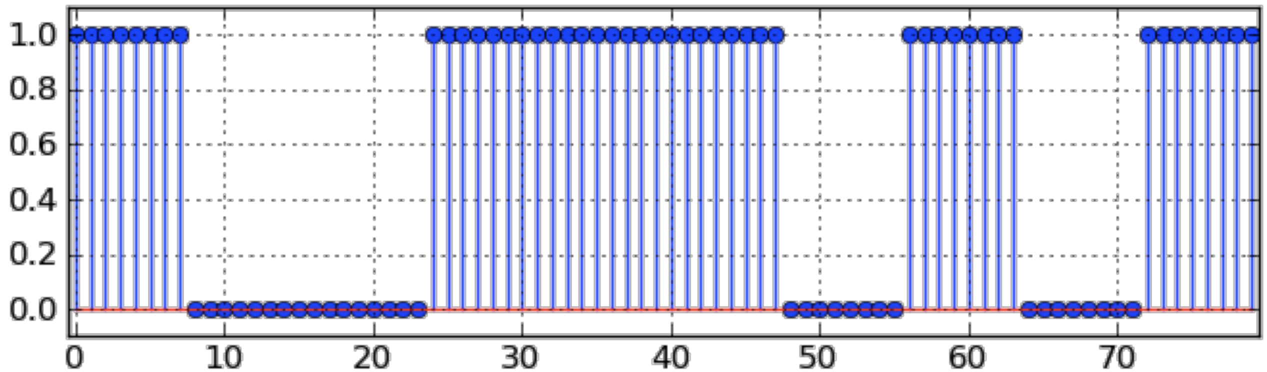
DIGITAL COMMUNICATION SYSTEMS

6.02 Fall 2014
Lecture #12

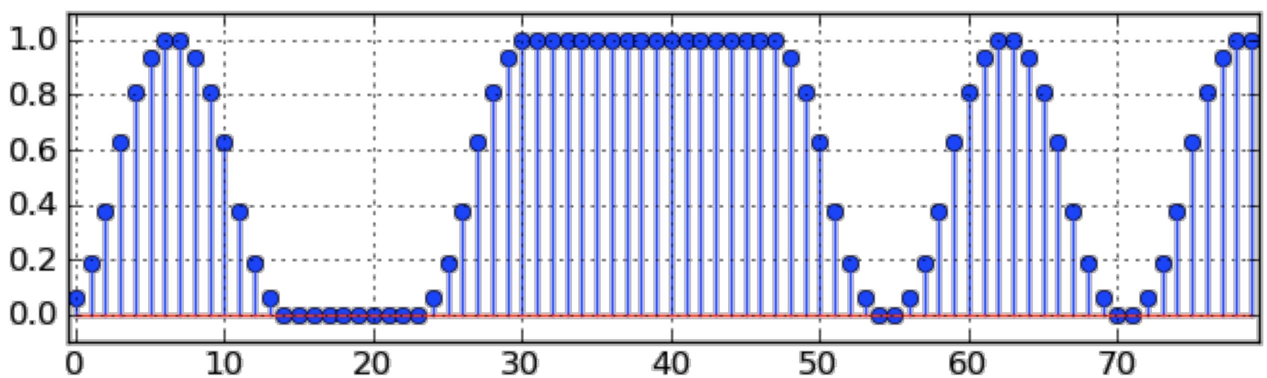
- Frequency response

Transmission Over a Channel

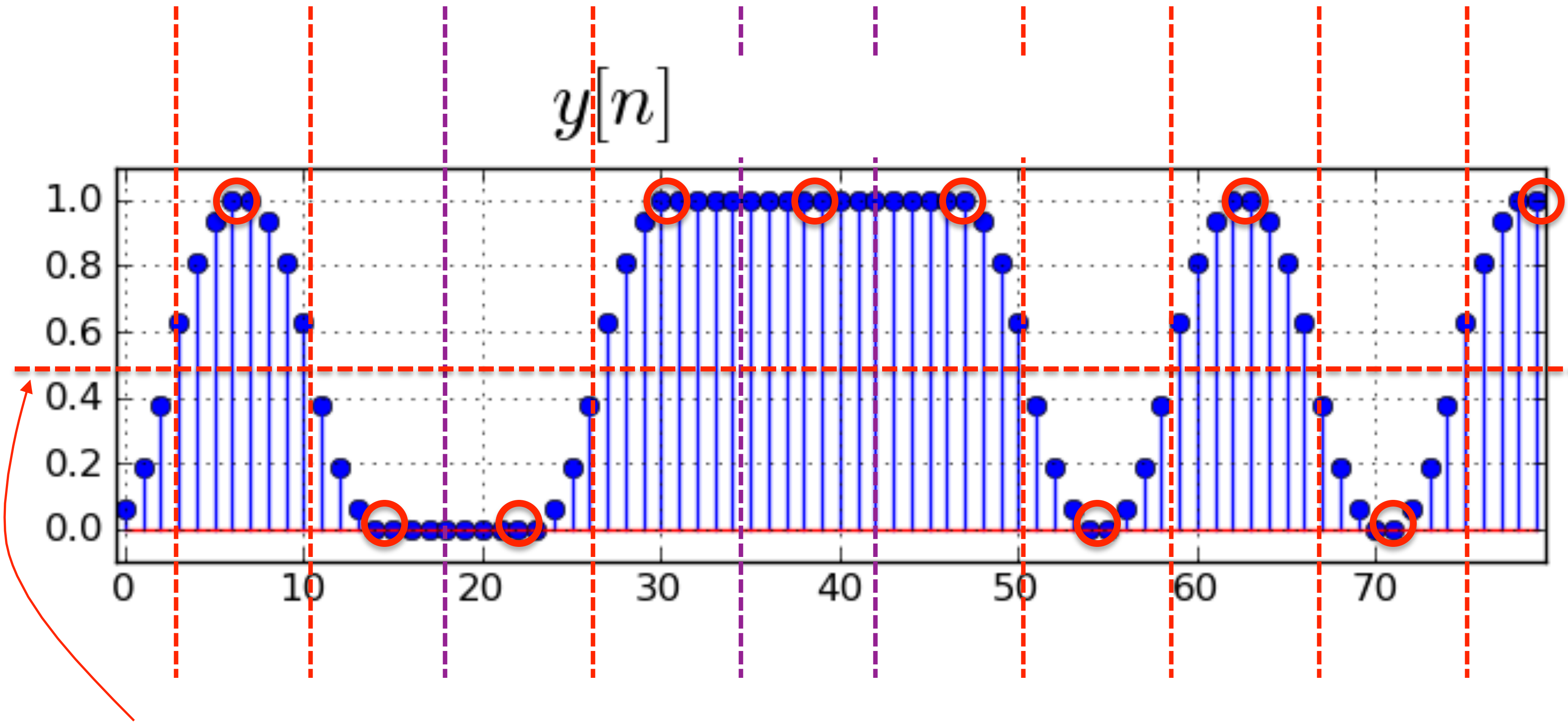
$x[n]$ at 8 samples/bit



$y[n]$

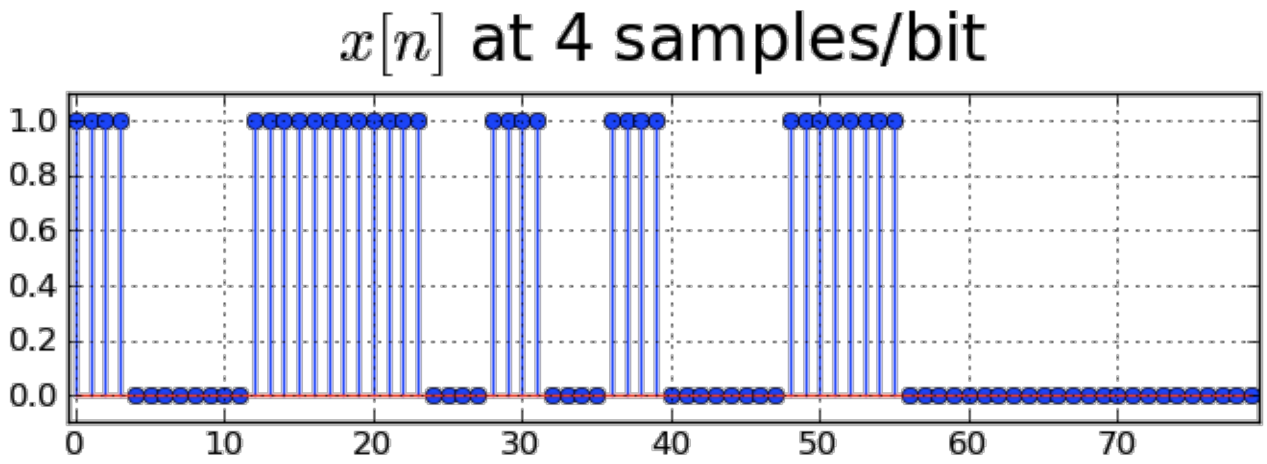


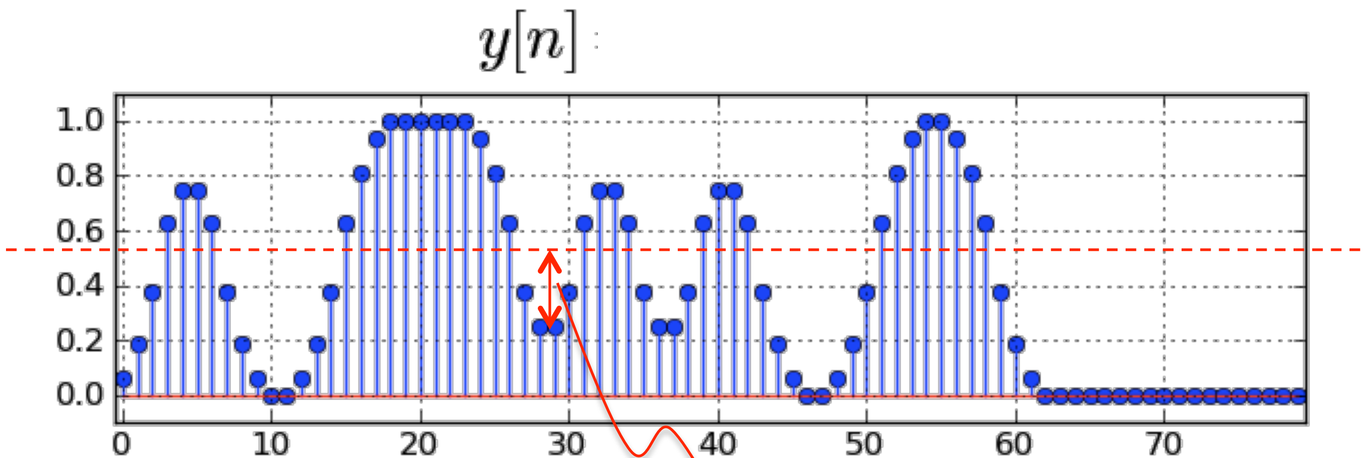
Receiving the Response



Digitization threshold = 0.5V

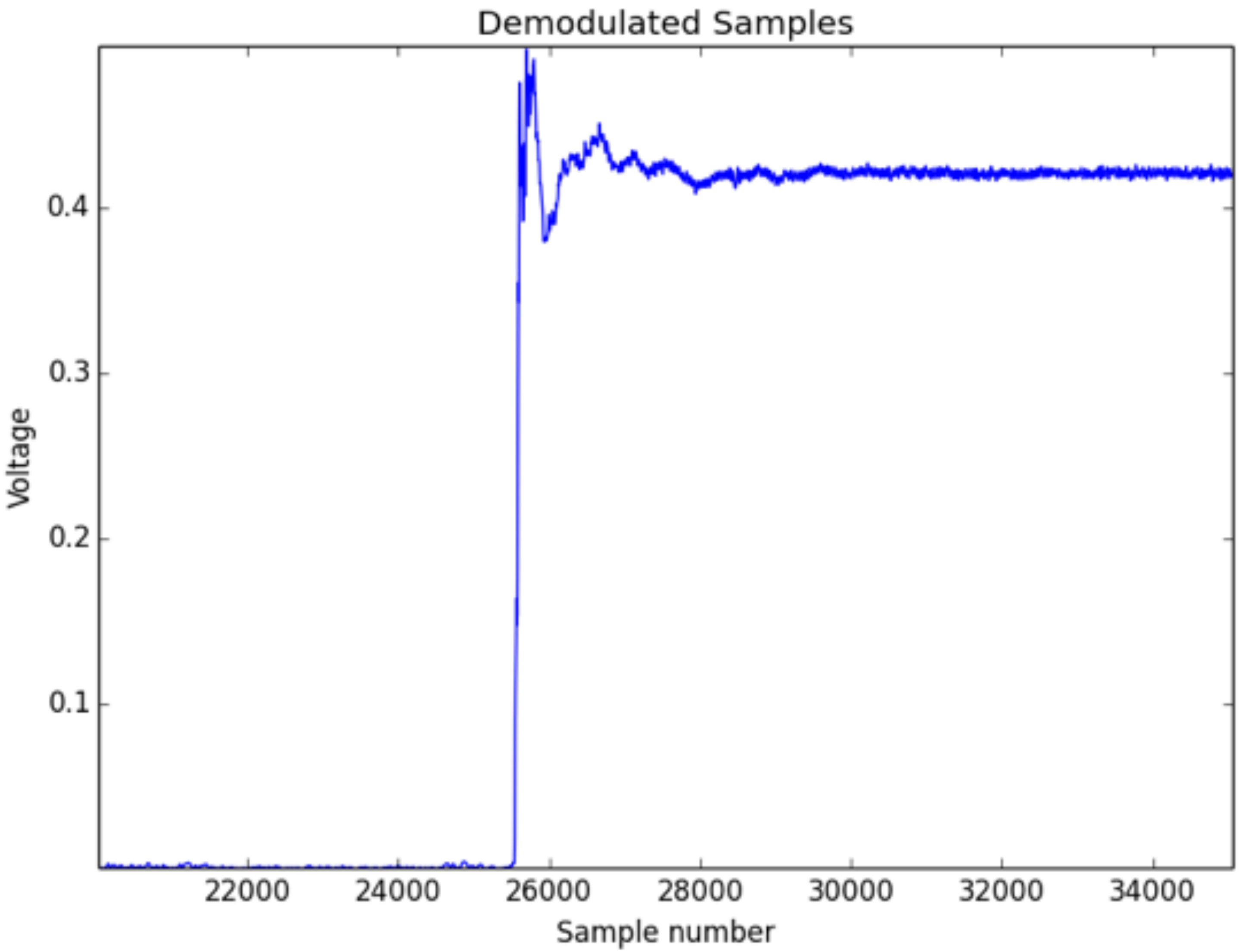
Intersymbol Interference





Noise margin? $0.5 - y[28]$

Step response of audiocom channel



Modeling LTI Systems

If system S is both linear and time-invariant (LTI), then we can use the unit sample response to predict the response to *any* input waveform x[n]:

Sum of shifted, scaled unit sample functions

Sum of shifted, scaled responses

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \boxed{S} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

CONVOLUTION SUM

Indeed, the unit sample response h[n] completely characterizes the LTI system S, so you often see

$$x[n] \longrightarrow \boxed{h[.]} \longrightarrow y[n]$$

To Convolve (not "Convolute"!)

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

A simple graphical implementation:

Plot x[.] and h[.] as a function of the dummy index (k or m above)

Flip (i.e., reverse) one signal in time,
slide it right by n (slide left if n is –ve), take the
dot.product with the other.

This yields the value of the convolution at the single time n.

'flip one & slide by n …. dot.product with the other'

Bounded-Input Bounded-Output (BIBO) Stability

What ensures that the infinite sum

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

is well-behaved?

One important case: If the unit sample response is *absolutely summable,* i.e.,

$$\sum_{m=-\infty}^{\infty} |h[m]| < \infty$$

and the input is *bounded,* i.e., $|x[k]| \leq M < \infty$

Under these conditions, the convolution sum is well-behaved, and the *output* is guaranteed to be *bounded*.

The absolute summability of h[n] is necessary and sufficient for this bounded-input bounded-output (BIBO) stability.

Time Now for a **Frequency-Domain** Story

in which
convolution
is transformed to
multiplication,
and other
good things
happen

A First Step

Do periodic inputs to an LTI system, i.e., x[n] such that

$$x[n+P] = x[n] \text{ for all } n, \text{ some fixed integer } P > 0$$

(with P usually picked to be the smallest positive integer for which this is true) yield periodic outputs? If so, of period P?

Yes! --- Since the system is TI, using input x delayed by P should yield y delayed by P. But x delayed by P is x again, so y delayed by P must be y. (Linearity is not needed.)

Alternate argument: use Flip/Slide/Dot.Product to see this easily: sliding by P gives the same picture back again, hence the same output value.

But much more is true for Sinusoidal Inputs to LTI Systems

Sinusoidal inputs, i.e.,

$$x[n] = \cos(\Omega n + \phi)$$

yield sinusoidal outputs at the same angular 'frequency' Ω rads/sample.

Note that such inputs are not even periodic in general.

Periodic if and only if $2\pi/\Omega$ is rational, =P/Q for some integers P(>0), Q. The smallest such P is the period.

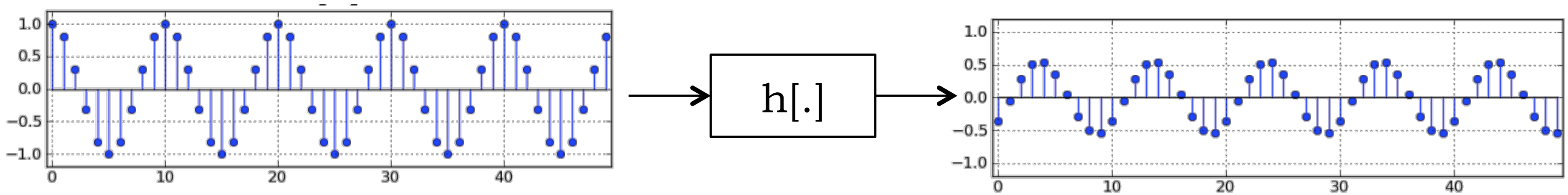
Nevertheless, we often refer to $2\pi/\Omega$ as the 'period' of this sinusoid, whether or not it is a periodic discrete-time sequence. This is the period of an underlying continuous-time sinusoid that, sampled at integer times, produces the given discrete-time sequence.

Examples

$\cos(3\pi n/4)$ has frequency $3\pi/4$ rad/sample, and period 8; shifting by integer multiples of 8 yields the same sequence back again, and no integer smaller than 8 accomplishes this.

$\cos(3n/4)$ has frequency ¾ rad/sample, and is not periodic as a DT sequence because $8\pi/3$ is irrational, but we could still refer to $8\pi/3$ as its 'period', because we can think of the sequence as arising from sampling the periodic continuous-time signal $\cos(3t/4)$ at integer t.

Sinusoidal Inputs and LTI Systems



A very important property of LTI systems or channels:

> If the input x[n] is a sinusoid of a given amplitude, frequency and phase, the response will be a *sinusoid at the same frequency*, although the amplitude and phase may be altered. The change in amplitude and phase will, in general, depend on the frequency of the input.

Let's prove this to be true … but use complex exponentials instead, for clean derivations that take care of sines and cosines (or sinusoids of arbitrary phase) simultaneously.

A related simple case: real discrete-time (DT) exponential inputs also produce exponential outputs of the same type

- Suppose $x[n] = r^n$ for some real number r
- $$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$
$$= \sum_{m=-\infty}^{\infty} h[m]r^{n-m}$$
$$= \left(\sum_{m=-\infty}^{\infty} h[m]r^{-m} \right) r^n$$
- i.e., just a scaled version of the exponential input

Complex Exponentials

Euler's formula shows the relation between complex exponentials and our usual trig functions:

$$e^{j\varphi} = \cos(\varphi) + j\sin(\varphi)$$

$$\cos(\varphi) = \frac{1}{2}e^{j\varphi} + \frac{1}{2}e^{-j\varphi} \qquad \sin(\varphi) = \frac{1}{2j}e^{j\varphi} - \frac{1}{2j}e^{-j\varphi}$$

In the complex plane, $e^{j\varphi} = \cos(\varphi) + j\sin(\varphi)$ is a point on the unit circle, at an angle of φ with respect to the positive real axis.
cos and sin are projections on real and imaginary axes, respectively.

Increasing φ by 2π brings you back to the same point!
So any function of $e^{j\varphi}$ only needs to be studied for φ in [-π, π] .

Useful Properties of $e^{j\varphi}$

When $\varphi = 0$:

$$e^{j0} = 1$$

When $\varphi = \pm\pi$:

$$e^{j\pi} = e^{-j\pi} = -1$$

$$e^{j\pi n} = e^{-j\pi n} = \left(-1\right)^n$$

(More properties later)

Frequency Response

$A(\cos\Omega n + j\sin\Omega n) = Ae^{j\Omega n}$ → h[.] → y[n]

Using the convolution sum we can compute the system's response to a complex exponential (of frequency Ω) as input:

$$
\begin{aligned}
y[n] &= \sum_m h[m]x[n-m] \\
&= \sum_m h[m]Ae^{j\Omega(n-m)} \\
&= \left(\sum_m h[m]e^{-j\Omega m}\right)Ae^{j\Omega n} \\
&= H(\Omega)\cdot x[n]
\end{aligned}
$$

where we've defined the *frequency response* of the system as

$$
\boxed{H(\Omega) \equiv \sum_m h[m]e^{-j\Omega m}}
$$

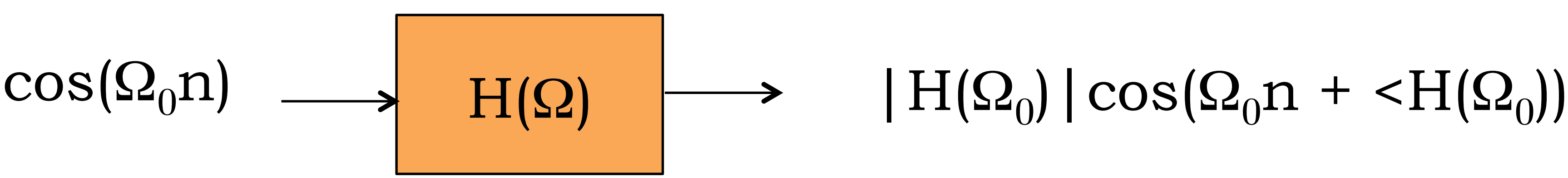
From Complex Exponentials to Sinusoids

$$\cos(\Omega n) = (e^{j\Omega n} + e^{-j\Omega n})/2$$

So response to this cosine input is

$$(H(\Omega)e^{j\Omega n} + H(-\Omega)e^{-j\Omega n})/2 = \text{Real part of } H(\Omega)e^{j\Omega n}$$

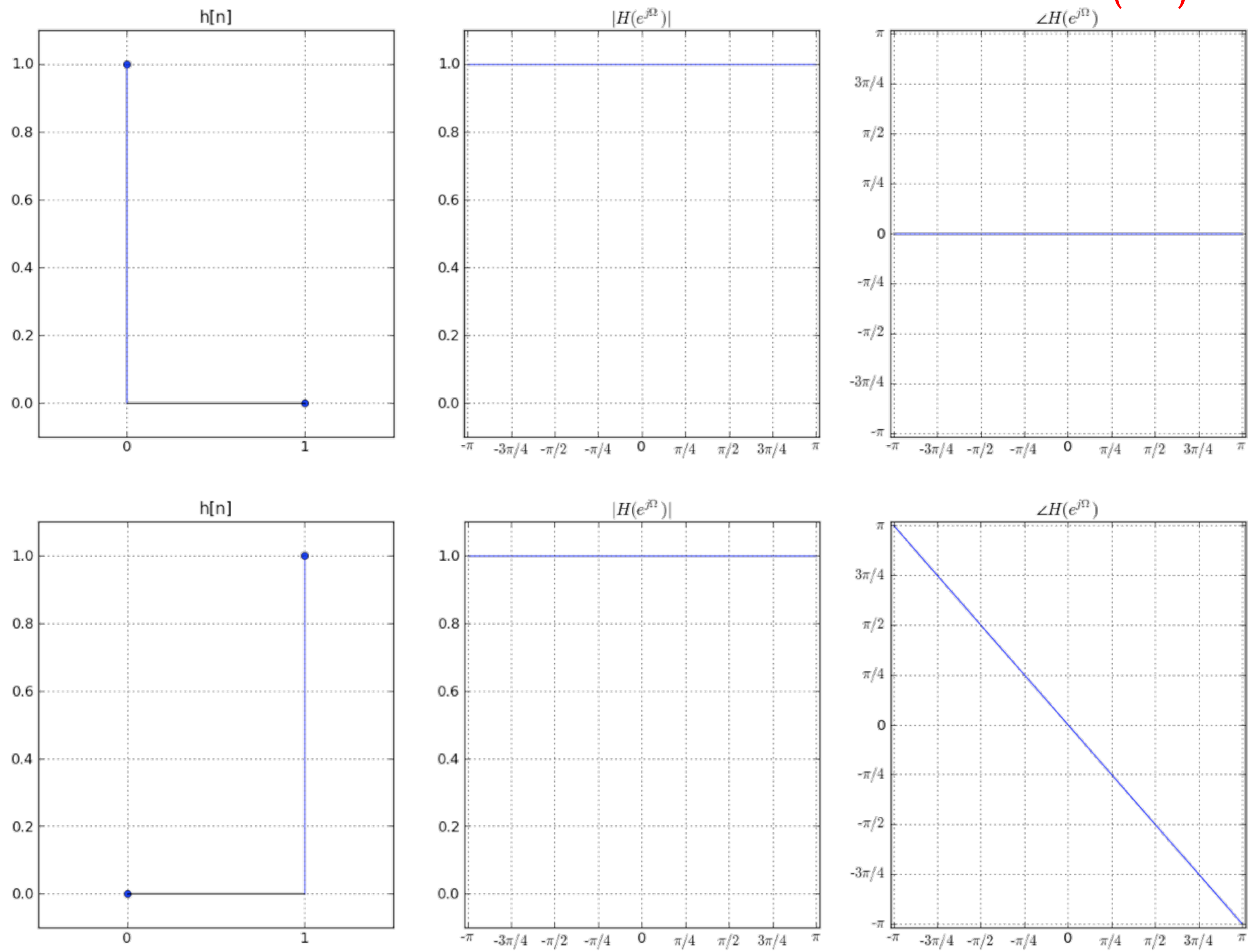
$$= \text{Real part of } |H(\Omega)|e^{j(\Omega n + <H(\Omega))}$$



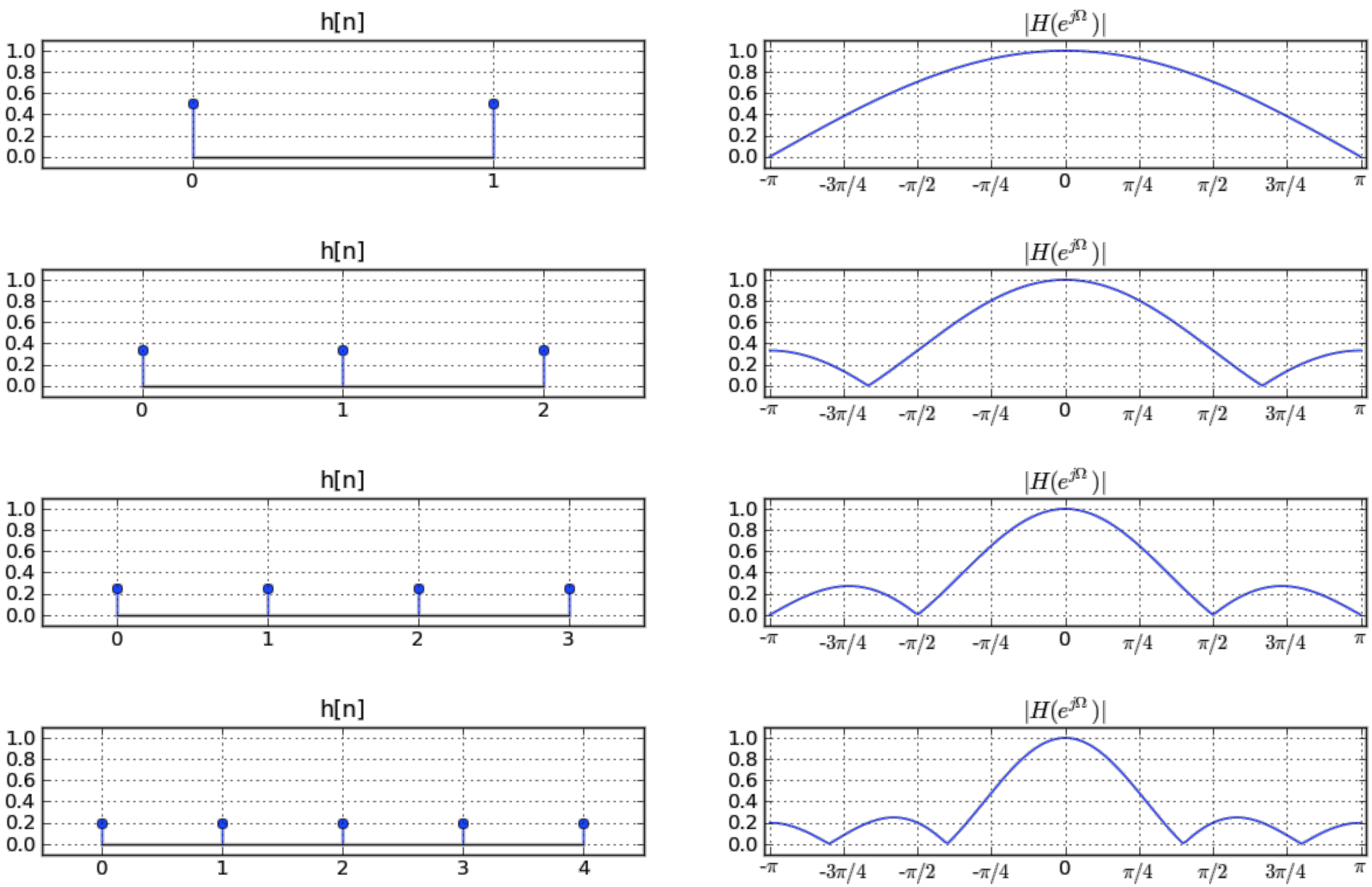
This is IMPORTANT

Example h[n] and H(Ω)

Sometimes written as $H(e^{j\Omega})$



Frequency Response of "Moving Average" Filters

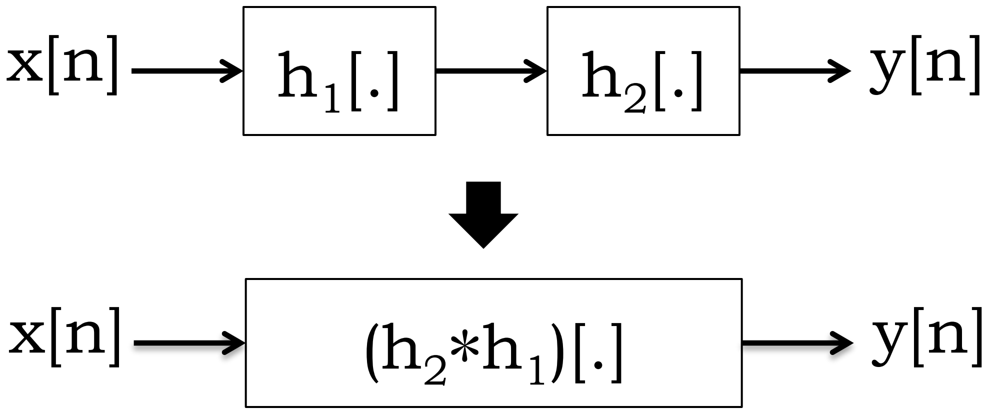


Relating Time- and Frequency-Domain Properties

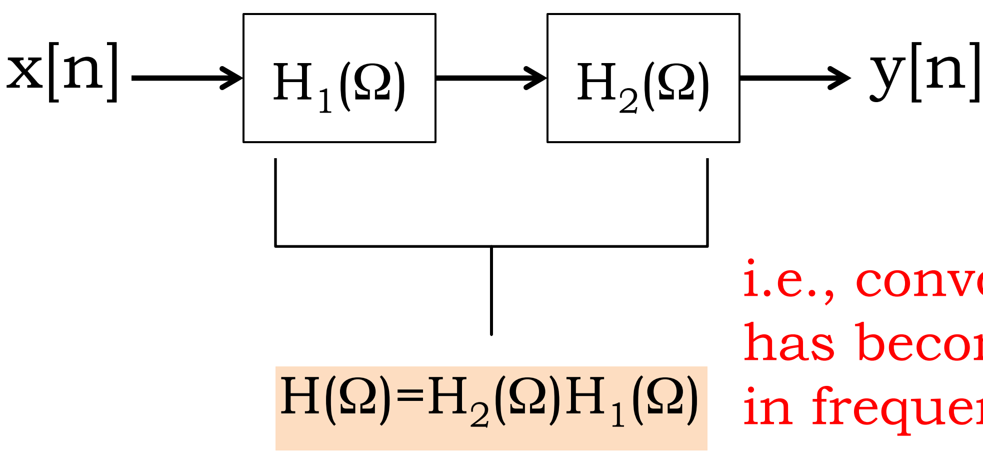
- Why does the 5-point moving average filter on the preceding page have nulls in its frequency response magnitude at $\pm 0.4\pi$ and $\pm 0.8\pi$?

(Think of convolving the unit sample response of this filter with sinusoids at these frequencies.)

Convolution in Time <---> Multiplication in Frequency

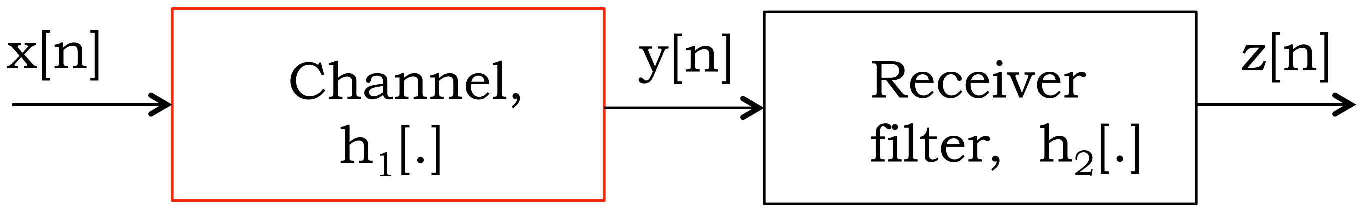


In the frequency domain (i.e., thinking about input-to-output frequency response):



i.e., convolution in time has become multiplication in frequency!

Example: "Deconvolving" Output of Channel with Echo



Suppose channel is LTI with

$$h_1[n]=\delta[n]+0.8\delta[n-1]$$

$$H_1(\Omega) = ?? \quad = \boxed{\sum_m h_1[m]e^{-j\Omega m}}$$

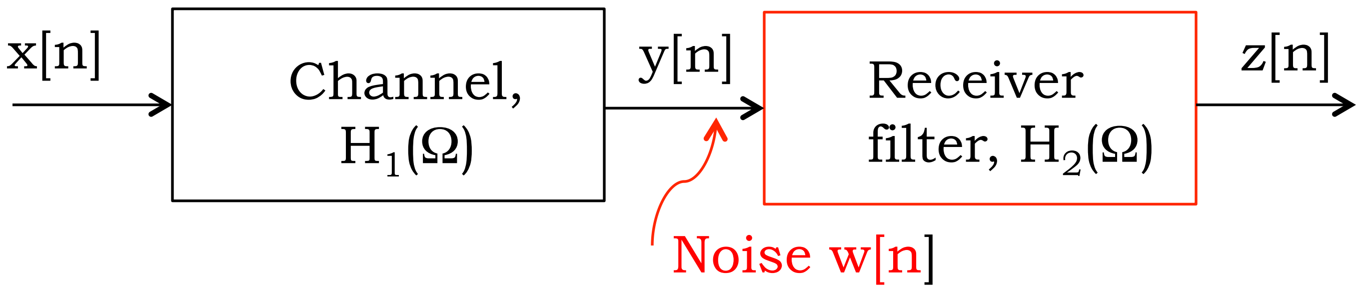
$$= 1 + 0.8e^{-j\Omega} = 1 + 0.8\cos(\Omega) - j0.8\sin(\Omega)$$

So:

$$|H_1(\Omega)| = [1.64 + 1.6\cos(\Omega)]^{1/2} \qquad \textit{EVEN function of } \Omega;$$

$$\angle H_1(\Omega) = \arctan\,[-(0.8\sin(\Omega)/[1 + 0.8\cos(\Omega)] \qquad \textit{ODD}.$$

A Frequency-Domain View of Deconvolution



Given $H_1(\Omega)$, what should $H_2(\Omega)$ be, to get z[n]=x[n]?

$$H_2(\Omega) = 1 / H_1(\Omega)$$ "Inverse filter"

$$= (1/|H_1(\Omega)|).\ \exp\{-j\angle H_1(\Omega)\}$$

Inverse filter at receiver can do very badly in the presence of noise that adds to y[n]:

filter has high gain for noise precisely at frequencies where channel gain $|H_1(\Omega)|$ is low (and channel output is weak)!

A Deeper Reason for Interest in Sinusoidal Inputs

- General inputs x[.] can be written as "sums" of sinusoids
- Each input sinusoidal component is mapped via the frequency response $H(\Omega)$ to its corresponding sinusoidal output component
- Superposition of these output components yields the general response y[.]

We'll develop this story over the next couple of lectures.